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Predicting the Price Effect of Mergers with Polynomial Logit Demand

ATANU SAHA and PETER SIMON

ABSTRACT *We propose a polynomial logit model to quantify the price effects of mergers in a static Nash setting. The proposed model is parsimonious in parameters and is shown to have excellent predictive power, rivaling the in-sample and out-of-sample predictive accuracy of the widely-used AIDS model. The analysis, using actual scanner data on bread sales, demonstrates that a linear logit model is likely to over-estimate the merger price effect.*

Key words: Mergers; Antitrust; Discrete choice; Logit.

JEL classification: L40, D43.

1. Introduction

In a number of papers on merger simulation, Werden and his co-authors (see Werden and Froeb, 1994; Werden, 1996, 1997a, b) have employed the logistic model of consumer demand. This paper extends the logit discrete choice model of consumer demand to an indirect utility function that is polynomial in price, with the linear specification being a special case. The polynomial specification allows the data to estimate the unknown curvature of demand. This is significant because the demand curvature plays a key role in determining the price effects of mergers (see Crooke *et al.*, 1999). Furthermore, the polynomial logit model, though parsimonious in parameters, is shown to have good predictive power, rivaling the in-sample and out-of-sample predictive accuracy of the widely-used Almost Ideal Demand System (AIDS) model (Hausman *et al.*, 1994; Hausman and Leonard, 1997).

Atanu Saha and Peter Simon are Principal and Associate with Analysis Group/Economic, respectively. They specialize in applied economics and statistical analysis, and have analyzed antitrust issues on behalf of various clients. They thank Luke Froeb, the anonymous referees, and participants at the session on Production Issues at the Western Economics Association Meetings for helpful comments without implicating them for remaining errors. The dataset used for this paper pertains to an actual merger in the bread industry but further details are kept confidential at the request of clients.
Atanu Saha, Analysis Group/Economics, 405 Lexington Av. 47th Floor, New York, NY 10174, USA; e-mail: asaha@analysisgroup.com.

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Though restrictive (Hausman and McFadden, 1984), the logit model has some advantages over the more general AIDS specification in a simulation context. The logit model constrains the cross-price elasticities to be positive, and the first-order conditions imply positive marginal costs.

The empirical application in the paper uses actual scanner data on bread sales and considers a hypothetical merger of two firms. The simulation results show that model specification does make a difference: in this case, the linear logit specification overstates the likely price effect of the merger.

The rest of the paper's structure is as follows: Section 2 presents the model; Section 3 presents estimation and simulation results; and Section 4 concludes.

2. The Model

Suppose there are n variants of a differentiated product sold at prices $p_1, \ldots, p_n$. Further assume that consumers' preferences are separable. The indirect sub-utility function of a consumer selecting the ith variant is assumed to be:

$$V_i = y + h(p_i) + \alpha_i + \varepsilon_i \tag{1}$$

where y denotes income, and $h(\cdot)$ is a continuous and decreasing function of p_i, α_i is the quality index of the ith variant, and ε_i is a random variable reflecting 'fluctuations in perception, attitudes or other non-measured factors' (McFadden, 1986: 278). The indirect utility function in (1) nests the linear form widely used in discrete choice modeling, $V_i = y - \beta \cdot p_i + \alpha_i + \varepsilon_i$, as a special case (see Anderson *et al.*, 1992, for an excellent review).

The consumer chooses the variant of the differentiated product that yields the greatest utility, that is, the purchase probability of the ith product is:

$$\tau_i = \Pr\{V_i = \max_{j=1,\ldots,n} V_j\}. \tag{2}$$

Assuming ε_i to be identically and independently distributed as a extreme value variate, the choice probability can be written as:

$$\tau_i(\mathbf{p}) = \frac{\exp(\alpha_i + h(p_i))}{\sum_{j=1}^{n} \exp(\alpha_j + h(p_j))} \tag{3}$$

where $\mathbf{p} \equiv \{p_1, \ldots, p_n\}$, (Anderson *et al.*, 1992). By construction, $\Sigma^n_{j=1} \tau_j(\mathbf{p}) = 1$ and $0 < \tau_i(\mathbf{p}) < 1$.

Typically, the n equations embodied in (3) can be estimated as a system of equations, where the quantity share of the ith good provides data on the market share. However, in what follows, we have developed the expressions for elasticities and the firms' first-order conditions in terms of revenue shares as opposed to quantity shares. The principal reason for adopting this approach is to compare the predictive power of the proposed polynomial logit model in (3) with that of the AIDS model, which is formulated in terms of revenue shares and not quantity shares. Furthermore, one can always recover the expression for quantity shares from the expression of revenue shares by replacing $h(p_i)$ with $h(p_i) - \log(p_i)$ in (3). Note that the development of the model in terms of revenue shares in the absence of an outside good implies that the aggregate price elasticity of demand is negative one.

Own and Cross Price Elasticities

Denoting $x_i(\mathbf{p})$ as the quantity demanded for the ith variant and assuming that

$$s_i = \frac{\exp(\alpha_i + g(p_i))}{\sum_{j=1}^{n} \exp(\alpha_j + g(p_j))}$$

denotes the revenue share of the ith variant, where $g(\cdot)$ is a decreasing function of p_i, we derive the formulas for own and cross-price elasticities as follows:

$$\eta_{ii}(\mathbf{p}) = \frac{\partial x_i(\mathbf{p})}{\partial p_i} \cdot \frac{p_i}{x_i} = \frac{\partial \ln s_i(\mathbf{p})}{\partial \ln p_i} - 1 = p_i \cdot g'(p_i) \cdot (1 - s_i) - 1, \text{ and}$$

$$\eta_{ij}(\mathbf{p}) = \frac{\partial x_i(\mathbf{p})}{\partial p_j} \cdot \frac{p_j}{x_i} = \frac{\partial \ln s_i(\mathbf{p})}{\partial \ln p_j} = - p_j \cdot g'(p_j) \cdot s_j.$$

It follows from the above that:

$$\eta_{ij}(\mathbf{p}) = -(1 + \eta_{jj}(\mathbf{p})) \cdot \frac{s_j}{1 - s_j}, \tag{4}$$

which demonstrates IIA restrictions on the cross-price and own-price elasticities, *i.e.* the cross-price elasticity of brand i with respect to brand j is the same for all i, and it is positively related to the absolute value of the jth brand's own price elasticity and its market share.

When $g'(p_i) < 0$ or when the consumer's indirect utility function is strictly decreasing in price, the logistic model implies that each brand's demand curve is negatively sloped and all brands are substitutes.

Equilibrium Conditions

We assume that there are K firms producing the differentiated product in question and each firm produces one or more brands/variants of the product, but no brand is produced by more than one firm. In particular, let m_k denote the number of variants produced by the kth firm so that

$$n = \sum_{j=1}^{k} m_j.$$

The kth firm's profit function is given by:

$$\pi^k = \sum_{i=1}^{m_k} (p_i - c_i) \cdot x_i(\mathbf{p}) - F_i \tag{5}$$

where $x_i(\mathbf{p})$ is the demand for good i, c_i is the constant marginal cost and F_i is the fixed cost of the ith variant. The firm's m_k first-order conditions are:

$$\frac{\partial \pi^k}{\partial p_i} = x_i(\mathbf{p}) + \sum_{j=1}^{m_k} (p_j - c_j) \cdot \frac{\partial x_j(\mathbf{p})}{\partial p_i} = 0,\ i = 1, \ldots, m_k. \quad (6)$$

Now, let the price-marginal cost margin be defined as

$$\theta_i = \frac{p_i - c_i}{p_i}$$

and use the expressions for η_{ii} and η_{ij} from above to rewrite the first-order conditions as:

$$\sum_{j=1}^{m_k} \theta_j \cdot \eta_{ji}(\mathbf{p}) \cdot \frac{s_j(\mathbf{p})}{s_i(\mathbf{p})} = -1,\ i = 1, \ldots, m_k \quad (7)$$

In the special case in which the firm produces only one variant of the product, the above first-order conditions reduce to the familiar single equation $\theta_i \cdot \eta_{ii} = -1$. The absence of an outside good implies that profit functions are concave which implies the existence and uniqueness of Nash equilibrium. In a number of more general model specifications (including the AIDS), the existence of equilibrium is not imposed by the functional form.

The Nash equilibrium prices, $\mathbf{p} \equiv \{p_1, \ldots, p_n\}$, are the solutions to the system of equations derived from the n first-order conditions of all K firms, expressed in matrix form as:

$$\mathbf{\Omega}_n \cdot \theta_n = -\mathbf{i}_n.$$

In (7), $\mathbf{\Omega}_n$ is an $n \times n$ block diagonal matrix with the matrices $(\Omega_1, \Omega_2, \ldots, \Omega_k)$ along its diagonal, where Ω_k is an $m_k \times m_k$ matrix whose $i-j^{\text{th}}$ element is

$$\eta_{ji} \cdot \left(\frac{s_j}{s_i}\right); \quad (8)$$

$\theta_n \equiv \{\theta_1, \ldots, \theta_k\}$ is an $n \times 1$ vector, and $\mathbf{i}_n$ is an $n \times 1$ vector of ones. Given observed prices, brand shares, and estimated elasticities (i.e. η_{ij}s), (8) can be solved for the brand-specific marginal costs:

$$c_i = p_i \cdot (1 - D_i), \quad (9)$$

where D_i denotes the ith element of $-\mathbf{\Omega}^{-1}{}_n \cdot \mathbf{i}_n$.

Now assume that firms 1 and 2 intend to merge. Then the first-order conditions of the merged entity are:

$$\frac{\partial \pi^{1+2}}{\partial p_i} = x_i(\mathbf{p}) + \sum_{j=1}^{m_1 + m_2} (p_j - c_j) \cdot \frac{\partial x_j(\mathbf{p})}{\partial p_i} = 0,\ i = 1, \ldots, m_1 + m_2. \quad (6')$$

The above equations and the unchanged first-order conditions of firms 3, . . ., K, form the post-merger system of equations which can be solved for the post-merger equilibrium prices.

Retail and Producer Prices

The profit functions are expressed in terms of producer prices, but the models are typically estimated using prices posted by retailers, which include the retailer's markup and so will be different from the producer prices. In the simulation experiments that follow, we assume that retail prices are 10% higher than producer prices.

Cost Efficiencies

In the model presented above, we can either solve for post-merger prices or marginal cost reductions that are sufficient to offset the price effects of the merger. This is done by inserting pre-merger prices into the post-merger first-order conditions and solving for the marginal costs that would be consistent with no price change. Since prices do not change, this can be done analytically, as was demonstrated by Werden (1996).

3. Estimation and Simulation Results

We estimate the model using weekly scanner data on eight brands of white pan bread (see Werden, this issue) quantities and prices in a US city for the period 23 January 1994 – 8 March 1998. Brands 1, 2 and 3 are produced by Firm A; brands 4, 5 and 6 are produced by Firms B, C and D, respectively; brand 7 is an aggregate of the grocery-store brands and brand 8 constitutes all remaining brands, each with less than 3% market share. We have used the scanner data to estimate the logit share equations.

Estimation Results

For the functional form of the choice probabilities in the polynomial logit specification, we have assumed $g(p_i) = \beta \cdot p_i + \gamma \cdot p^2_i/2$, which can be viewed as a second-order Taylor series approximation to any true but unknown function $g(\cdot)$. This specification nests the linear logit as a special case when $\gamma = 0$.

However, there is no *a priori* reason to restrict the functional form for $g(\cdot)$ to be a second-order polynomial in price (see, for example, Crooke *et al.*, 1999 for discussion of the curvature properties of various models). In fact, we have estimated the model with higher order terms in $g(\cdot)$, and found that these do not yield any significant improvement in predictive accuracy, i.e. the estimated coefficients of these higher-order terms were not statistically significant. We adopt the normalization

$$\sum_{j=1}^{n} \alpha_j = 1$$

without loss of generality. The estimated parameters are presented in Table 1.

Table 1. Estimation results

Parameter	Parameter estimates	
	Estimate	Asymptotic t-ratio
α_1	0.29	12.50
α_2	−0.31	−8.09
α_3	−0.08	−4.82
α_4	−0.22	−13.78
α_5	0.05	1.57
α_6	0.04	1.43
α_7	0.92	29.65
α_8	0.32	12.09
β	0.09	0.68
γ	−0.32	−4.22

Number of observations: 219 weekly observations on each firm.
Correction for first-order autocorrelation has been made, using a different autocorrelation coefficient for each equation.

Of the 213 observations in the data set, 23 observations (approximately 10%) were set aside to test the out-of-sample predictive power of alternative models. Seven revenue-share equations were estimated as a system using nonlinear least squares, with and without correction for serial correlation of the errors. In both cases, the linear version, $g(p_i) = \beta \cdot p_i$, was rejected (the asymptotic t-statistics for γ were −4.22 and −4.68).

The in-sample and out-of-sample root mean squared errors (RMSE), averaged across the seven share equations, are reported in Table 2. Comparison of the RMSE of the linear and ploynomial logit specifications reveals that the polynomial form performs considerably better both in-sample and out-of sample.

We have also estimated the widely-used AIDS model (Deaton and Muellbauer, 1980) as a benchmark. The difference between the average RMSE for the AIDS and polynomial logit models is around 1% both within sample and out-of-sample. This is remarkable considering there are 42 parameters in the AIDS model versus only nine in the polynomial logit model. Elasticity matrices for the AIDS and nonlinear

Table 2. Average root mean squared errors for alternative models

	Multinomial logit model		Percentage difference	AIDS model	Percentage difference
	Linear	Polynomial			
	[A]	[B]	([A] − [B])/[A]	[C]	([B] − [C])/[C]
In-sample	0.0161	0.0116	28%	0.0115	1.12%
Out-of-sample	0.0254	0.0155	39%	0.0154	0.68%

Table 3. Comparison of AIDS and polynomial logit elasticity matrices

AIDS Brands	1	2	3	4	5	6	7	8
1	−1.42	0.09	0.04	0.04	0.14	−0.05	0.15	−0.01
2	0.12	−1.64	0.21	0.05	0.06	0.26	−0.09	−0.18
3	0.07	0.24	−1.75	0.03	0.24	0.16	−0.04	0.01
4	0.08	0.07	0.03	−1.55	0.12	0.24	0.03	0.03
5	0.28	0.08	0.26	0.14	−2.30	0.25	0.13	0.12
6	−0.13	0.24	0.15	0.26	0.22	−1.68	0.09	−0.36
7	0.07	0.00	−0.01	0.00	0.03	0.04	−1.25	0.12
8	0.03	−0.06	0.03	0.04	0.08	−0.14	0.32	−1.03
Polynomial logit Brands	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	−1.49	0.04	0.08	0.04	0.11	0.10	0.06	0.07
2	0.08	−1.48	0.08	0.04	0.11	0.10	0.06	0.07
3	0.08	0.04	−2.03	0.04	0.11	0.10	0.06	0.07
4	0.08	0.04	0.08	−1.44	0.11	0.10	0.06	0.07
5	0.08	0.04	0.08	0.04	−2.46	0.10	0.06	0.07
6	0.08	0.04	0.08	0.04	0.11	−2.28	0.06	0.07
7	0.08	0.04	0.08	0.04	0.11	0.10	−1.12	0.07
8	0.08	0.04	0.08	0.04	0.11	0.10	0.06	−1.39

models are given in Table 3. Note, in the AIDS model, for five of the eight brands at least one cross-price elasticity is negative, implying that those brands are complements instead of substitutes.

Table 4 contains the estimates of elasticities and marginal costs for the polynomial logit model. In computing the marginal costs we have used (9), with the

Table 4. Estimated elasticities, marginal costs, and margins

Firm	Brand	Elasticities				Cost	Pre-merger	
		Own price estimate	Asymtotic t-ratio	Cross price estimate	Asymtotic t-ratio	Marginal cost	Prices	Margin %
A	1	−1.37	−31.13	0.06	9.92	$ 0.30	$ 1.30	76.56
A	2	−1.36	−28.55	0.03	8.92	0.27	1.24	78.14
A	3	−1.80	−24.72	0.07	12.45	0.65	1.78	63.67
B	4	−1.34	−28.07	0.03	8.40	0.31	1.21	74.77
C	5	−2.14	−17.58	0.10	10.38	1.12	2.10	46.68
D	6	−2.01	−19.78	0.09	11.08	1.00	1.98	49.73
Grocery store	7	−1.09	−27.51	0.04	2.86	0.07	0.79	91.65
All others	8	−1.29	−28.66	0.05	7.74	0.27	1.18	77.40

Note: The elasticities and marginal costs reported in this table have been computed using prices at the firm level, while the elasticities presented in Table 3 have been computed using retail-level prices.

Table 5. Simulation results

Pre-merger				Post merger			
Firm	Brand	Price	Market share	Polynomial logit			Linear logit
				Firm	% Price change from pre-merger	Compensating marginal cost reduction	% Price change from pre-merger
A	1	$1.30	14.2%	A + B	3.5%	−10.1%	4.9%
A	2	1.24	8.0%	A + B	3.4%	−10.1%	4.9%
A	3	1.78	7.6%	A + B	4.2%	−10.1%	5.2%
B	4	1.21	8.8%	A + B	11.0%	−25.1%	16.6%
C	5	2.10	7.0%	C	0.0%	0.0%	0.0%
D	6	1.98	7.6%	D	0.0%	0.0%	0.1%
Grocery store	7	0.79	31.5%	Grocery store	0.1%	0.0%	0.3%
All others	8	1.18	15.2%	All others	0.0%	0.0%	0.1%
Price change of all brands					**2.2%**		3.2%
Price change of merging firms' brands					**5.2%**		7.4%

Note: We have compared the pre-merger share-weighted price before and after the merger in computing the percentage price changes for all brands and merging firms' brands.

elasticities computed using the estimated demand parameters. The estimated own and cross-price elasticities and the brand-specific marginal costs suggest a high degree of heterogeneity across firms and brands. Notice that for most of the brands, a larger (in absolute value) own-price elasticity is associated with a large positive cross-price elasticity.

Simulation Results

We have considered the hypothetical merger of firms A and B; thus we have assumed that the merged entity would be producing brands 1–4. Note that the absence of an outside good in the logit demand implies that consumers' total expenditure on bread remains unchanged after the merger. Table 5 contains the pre-merger and post-merger shares and prices computed using the polynomial and linear versions of the logit model. Consistent with the high degree of heterogeneity in brand elasticities, the price effect of the proposed merger varies considerably across brands, ranging from no price change to an 11.0% increase. Not unexpectedly, the simulation results reveal that differences in model specification generate appreciably different estimates of the price effect of mergers. The linear logit model suggests that the proposed merger is likely to raise the average price of the merging firms' brands by 7.4%, whereas the corresponding estimate from the polynomial logit model is only 5.2%. In fact, the linear logit predicts higher prices post-merger for every brand relative to the polynomial logit predictions.

Cost Implications

We have also solved for the reduction in merging firms' marginal costs sufficient to offset the price effects of the merger. These results are also presented in Table 5. The compensating marginal cost reductions are about twice the predicted price rises.

4. Concluding Comments

This paper proposes a simple model to predict the price effects of a proposed merger. The estimation framework is more general than the linear logit, yet it is still easy to estimate. In particular, our specification lets the data determine the curvature of demand. Our results show that a more-restrictive specification, with an assumed curvature, would have over-estimated the merger price effect.

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