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Atanu Saha, *Compass Lexecon*
Burton G. Malkiel, *Princeton University*

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Abstract

We extend the analytical framework of traditional DCF models to allow for the possibility of a time-varying cessation risk for cash flows. We first set out a parsimonious functional form for time-dependent survival probability of cash flows and then derive a closed-form solution for cessation risk-adjusted discount rates within a DCF model. Application of the model to a new data set, created for this paper, demonstrates that U.S. start-up firms face considerable risk of cessation, particularly during the first five years of their existence. This finding suggests that the time-varying discount rates that are appropriate to value them are considerably higher than those used in traditional DCF models.

KEYWORDS: DCF, valuation, cessation risk

Author Notes: Atanu Saha, Senior Vice President and Head of NY Office, Compass Lexecon, 156 West 56th Street, 19th Floor, New York, NY 10019, USA; Tel: 212-782-3501; email: asaha@compasslexecon.com.
Burton G. Malkiel, Chemical Bank Chairman's Professor of Economics, Emeritus & Senior Economist, Princeton University, 26 Prospect Avenue, Princeton, NJ 08544, USA; Tel: 609-258-6445; email: bmalkiel@princeton.edu.
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1. INTRODUCTION

A fundamental principle of valuation states that the worth of any asset, be it a stock, bond, real estate property, etc., is the discounted present value of the stream of cash flows the asset is expected to produce. In many valuation settings, it is eminently reasonable to assume that cash flows have a non-zero probability of cessation. We demonstrate that if one allows for even a small probability that the cash flows will cease to exist, then a substantially higher discount rate is required in deriving the present value of the expected stream of cash flows.

In typical discounted cash flow (DCF) models, the discount rate is taken to be the base risk-free rate (the rate of interest on safe government securities) plus some risk premium. Most investment texts such as Ross et al. (2009), Brealey and Myers (2003), and Bodie et al. (2010) recommend that the appropriate discount rate be selected by reference to the Capital Asset Pricing Models (CAPM). Because CAPM-based discount rates only account for market risk, valuation models may greatly underestimate the discount rate (or overstate the net present value, NPV) in settings where the idiosyncratic risk of the cash flows matters. This is especially so in cases where there is a significant probability that the future stream of cash flows may completely cease. This is a risk that the CAPM ignores because that model assumes it is a risk that can be diversified away. While CAPM-based discount rates can provide a useful *starting point*, we believe that an additional adjustment to the discount rate is warranted to account for cash flow cessation probability, in settings where such a possibility is not immaterial.

In this paper, we first derive, in an infinite horizon setting, the closed-form solution for NPV, when cash flows have a finite probability of cessation at each period, and then present a simple formula for the cessation risk-adjusted discount rate. We then extend the analytical framework to allow for the possibility of a time-varying cessation risk, which we believe more realistically describes the pattern of failures for many types of enterprises, especially new ventures. Here, we first set out a parsimonious functional form for the time-varying survival probability of cash flows and then derive a closed-form solution for cessation risk-adjusted discount rate within a DCF model framework. Our results show that in a setting where the cessation risk declines rapidly over time, usage of a constant time-invariant cessation rate assumption yields an erroneous and artificially low valuation of the enterprise. On the other hand, ignoring cessation risk when such risk exists can greatly overstate the NPV.

Valuation practitioners have generally recognized the need for a discount rate that properly accounts for the higher riskiness in certain industries, particularly with start-up companies (see Thomas and Gup, 2010, for an excellent review). For example, a study undertaken by Sahlman and Scherlis (2009) at the Harvard Business School shows that venture capital companies value their investments in target companies using a very high discount rate, typically 35% or higher. The study argues that venture capitalists use these high discount rates to account for the material possibility that a venture will not succeed, and, as a result, that future cash flows would be zero. Similarly, investors in biotechnology companies face the possibility that an investment could quickly lose all of its value if a drug does not perform well in clinical trials or is not approved by regulators. A study on valuing biotechnology firms finds that biotechnology investors typically value investments using discount rates of 25% and higher (Frei and Leleux, 2004).

Hall and Woodward (2010) provide an excellent discussion of the challenges facing start-up firms, focusing on the information technology and biotechnology industries. The authors conduct a cohort study on venture-backed firms and analyze the distribution of both the firms' venture lifetime and exit status. Their findings show that nearly 65% of the venture-backed companies studied fail to achieve either an IPO, an acquisition or other successful exit. They also find that more than 15% of the firms cease to exist. Knaup (2005) analyzes business survival characteristics of U.S. firms from 1998 to 2002. Her findings show that on a national level, 80% of small businesses survive their first year; 65% survive their second year, and only 55% survive their third year.

While the applied finance literature has generally recognized the need for adjusting upward the discount rate when cash flows have a non-trivial probability of ceasing to exist, we believe ours is the first paper to set out an analytical framework to explicitly model the cessation risk-adjusted discount rate for a DCF model. It also sets out an empirical model to estimate cessation probability. An application using data on attrition rates of start-up firms in the U.S. illustrates our approach.

2. THE ANALYTICAL FRAMEWORK

In this section, we begin by briefly reviewing the DCF model framework with constant (i.e., time-invariant) cash flow cessation risk. We then set out the functional form for a time-dependent survival probability of cash flows. Finally, we incorporate this functional form in a DCF framework to derive the closed form solutions for both the expected NPV and the cessation risk-adjusted discount rate.

2.1 Time-Invariant Cessation Risk

In the analysis that follows, we assume, without loss of generality, that the cash flow at the beginning of the valuation period is \$1. We adopt the following notations: g = the growth rate of cash flows; r = the discount rate. Throughout the paper, we will assume that r is the discount rate, estimated using CAPM or a similar model, and does not account for cessation risk. Then NPV of the stream of cash flow is:

$$V_1 = \sum_{i=1}^{\infty} \left(\frac{1+g}{1+r} \right)^i = \frac{(1+g)}{(r-g)}. \tag{1}$$

DCF models to value common stocks date back at least to Fisher (1961) and Williams (1938). The simplest form of the model, as presented above, was popularized by Gordon (1959), as shown below:

$$r = \frac{(1+g)}{V_1} + g. \tag{2}$$

The rate of return can be obtained by adding the dividend yield (based on next year's dividend) to the growth rate.

Now assume, in addition to above, that at *each* period there is a finite probability (denoted by d) that the cash flow will cease to exist. Therefore, at any period, the probability of the continuation of cash flow is: $0 \le (1-d) \le 1$. In this framework, the NPV can be computed as:

$$V_2 = \sum_{i=1}^{\infty} (1-d)^i \left(\frac{1+g}{1+r} \right)^i = \frac{(1-d)(1+g)}{(r-g+d+d \cdot g)}. \tag{3}$$

It is clear from the comparison of (1) and (3) that V_2 nests V_1 as a special case when $d = 0$.

One can solve for the cessation probability-adjusted discount rate (denoted by r^*) within the traditional DCF framework by setting V_2 equal to V_1. That is, setting:

$$\frac{(1+g)}{(r^*-g)} - \frac{(1+g)(1-d)}{(r-g+d+d \cdot g)} = 0, \tag{4}$$

we get:

$$r^* = \frac{d+r}{(1-d)}.$$

Thus, r^* indicates the discount rate one should be using within the traditional DCF framework to calculate NPV, if one recognizes a finite but fixed probability of cash flow cessation.

2.2 Time-Varying Cessation Risk

In the preceding sub-section, we had assumed that at any time period the survival probability of the cash flow is time-invariant and equal to $(1-d)$. We now relax this assumption and set out a more general framework which subsumes the time-invariant probability structure as a special case.

Let the cash flow's survival probability in time period i be denoted by:

$$s(i) = 1 - \lambda_0 e^{-\lambda \cdot i}. \quad (5)$$

Thus, the time-varying cessation probability is: $d(i) = 1 - s(i) = \lambda_0 e^{-\lambda \cdot i}$.

Under the parameter restrictions $1 > \lambda_0 \geq 0$ and $\lambda \geq 0$, the survival probability in (5) has the following properties:

(a) $0 \leq s(i) \leq 1$, i.e., it satisfies the conditions for being a probability function.
(b) $s(i) = 1$ if $\lambda_0 = 0$, i.e., the traditional DCF model without cessation risk is a special case.
(c) $s(i) = 1 - \lambda_0$ if $\lambda = 0$, i.e., the time-invariant cessation rate of λ_0 is a special case.
(d) $\dfrac{\partial s(i)}{\partial i} > 0$ and $\dfrac{\partial^2 s(i)}{\partial i^2} < 0$ if $\lambda_0 > 0$ and $\lambda > 0$, i.e., the survival probability is a strictly increasing concave function of time.
(e) $s(i)$ changes at a constant rate of λ per period because $\dfrac{\partial s(i)}{\partial i} / s(i) = \lambda.$

The survival probability functional form in (5) can be readily estimated using ordinary least squares. Since $ln[1 - s(i)] = ln[d(i)] = ln(\lambda_0) - \lambda \cdot i$, the estimation equation can be written as:

$$y_t = \beta_0 + \beta \cdot T_t + \varepsilon_t, \quad (6)$$

where $y_t = ln[1 - s(t)]$, T_t is time trend variable taking values of 1, 2, …, n, denoting time periods of existence, and ε_t is the error term. Thus,

$$\hat{\lambda} = -\hat{\beta} \text{ and } \hat{\lambda}_0 = e^{\hat{\beta}_0 + \frac{\hat{\sigma}^2}{2}},$$

where $\hat{\sigma}$ is the root-mean square of the regression.[1] Such an estimation should enable us to determine if time-variant cessation rates are appropriate. If the estimation results fail to reject the null hypothesis $\beta = 0$, then one should be using the estimated time-invariant cessation rate of $\hat{\lambda}_0$ to compute the discount rate, i.e.,

$$r^* = \frac{\left(r + \hat{\lambda}_0\right)}{\left(1 - \hat{\lambda}_0\right)}.$$

2.3 DCF Model with Time-Varying Cessation Risk

With time-dependent survival probability set out in (5), the NPV within a DCF model is:

$$V_3 = \sum_{i=1}^{\infty} (1 - \lambda_0 e^{-\lambda \cdot i}) \left(\frac{1+g}{1+r}\right)^i = \frac{(1+g) \cdot [1 + g - e^{\lambda}(1+r) + \lambda_0 (r-g)]}{(r-g) \cdot [1 + g - e^{\lambda}(1+r)]}. \qquad (7)$$

It is readily verified that V_3 nests the traditional DCF framework as a special case, that is,

$$V_3 = \frac{(1+g)}{(r-g)} = V_1,$$

when $\lambda_0 = 0$. Setting V_1 equal to V_3, one can solve for the cessation risk-adjusted discount rate:

[1] The expression $\hat{\lambda}_0 = e^{\hat{\beta}_0 + \frac{\hat{\sigma}^2}{2}}$ arises from the normality assumption of the error term, ε_t. Under this assumption, $\hat{\beta}_0$ is normally distributed; hence, $\hat{\lambda}_0$ is log normally distributed with an expected value of $e^{\hat{\beta}_0 + \frac{\hat{\sigma}^2}{2}}$.

$$\hat{r} = \frac{g^2\lambda_0 - g(1+\lambda_0)\cdot r + r\cdot[e^{\lambda}(1+r)-1]}{g\cdot(\lambda_0 - 1) - \lambda_0\cdot r + [e^{\lambda}(1+r)-1]}. \quad (8)$$

Note that when $\lambda_0 = 0$,

$$\hat{r} = \frac{r\cdot\{[e^{\lambda}(1+r)-1]-g\}}{\{[e^{\lambda}(1+r)-1]-g\}} = r,$$

which is the cessation risk-unadjusted discount rate. Also, $\hat{r}$, unlike r^*, is not independent of g, the cash flow growth rate.

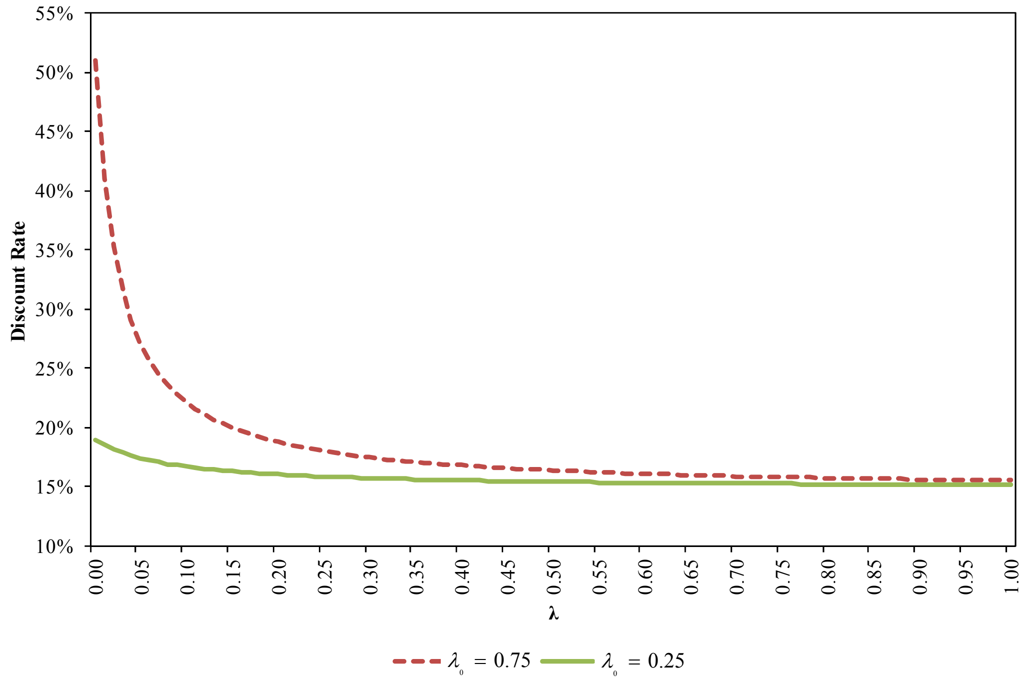
With estimates of λ_0 and λ from empirical data, we can then solve for $\hat{r}$ using (8). This provides the appropriate discount rate one should be using within the traditional DCF framework to calculate NPV, recognizing the time-varying probability of cash flow cessation.

It is instructive to observe how $\hat{r}$ changes with various values of λ and λ_0. Since higher values of λ imply a higher rate of decline of cessation rate, $\hat{r}$ is a decreasing function of λ. In fact,

$$\lim_{\lambda \to +\infty} \hat{r} = r. \quad (9)$$

By contrast, higher values of λ_0 imply higher cash flow cessation rate; thus, $\hat{r}$ is strictly increasing in λ_0. As a result, $\hat{r}$ approaches r as λ_0 gets smaller, and, as noted above, $\hat{r}$ is equal to r when $\lambda_0 = 0$. In Figure 1, we illustrate the sensitivity of $\hat{r}$ to various values of λ_0 and λ. In this figure, we have assumed $r = 15\%$ and $g = 3\%$. The values of $\hat{r}$ corresponding to various values of λ are depicted for $\lambda_0 = 0.25$ and for $\lambda_0 = 0.75$. It is evident from Figure 1 that: (a) $\hat{r}$ rapidly approaches r for $\lambda > 0.5$, regardless of the value of λ_0, and (b) the difference between cessation risk-adjusted ($\hat{r}$) and the unadjusted discount rate (r) can be substantial when λ_0 is large and λ is small. For example, if $\lambda = 0.05$, $\hat{r}$ is 17% when $\lambda_0 = 0.25$; however, when $\lambda_0 = 0.75$, $\hat{r}$ is 27%, substantially higher than the unadjusted discount rate of 15%.

Figure 1: Risk-Adjusted Discount Rates for Various Values of λ_0 and λ



In Table 1, we illustrate the implications for valuation of the proposed model framework using a simple numerical example and contrast the results with the two alternatives: no cessation risk and constant cessation risk. In Table 1, Scenario A is the traditional CAPM-based framework of no cessation risk. Here – as is the case in all three scenarios – we assume initial free cash flow of $100, per-period cash flow growth rate of 3%, and CAPM-based (i.e., without cessation risk) discount rate of 15%. Using the expression in (1), the NPV of this stream of cash flow over the infinite horizon is found to be $858.

Table 1: Parameter Estimates' Implications for Valuation

		Discount Rate	NPV	% Difference
Scenario A	No cessation risk	15.0%	$858	
Scenario B	Constant cessation risk of 10%	27.8%	$416	-51.6%
Scenario C	Time-varying cessation risk			
	$\lambda_0 = 0.75 \quad \lambda = 0.25$	18.0%	$685	-20.2%
	$\lambda_0 = 0.75 \quad \lambda = 0.95$	15.6%	$819	-4.6%

In Scenario B, we assume a constant cessation risk of 10% but keep the remaining assumptions unchanged. Using the expression that follows after (4), we find that the value of $d = 0.1$ implies the constant cessation risk-adjusted discount rate of $r^* = 27.8\%$. Thus, using (1), but using this r^* instead of r, we find that the NPV is \$416, which is 51.6% lower than the NPV, in Scenario A, without cessation risk.

In Scenario C, we discuss the time-varying cessation risk model's implications. We consider two alternative values of λ, reflecting different rates of decline of cessation risk. In both cases, however, we assume the same value of $\lambda_0 = 0.75$. Parenthetically, the parameter values assumed in Table 1 are largely consistent with our empirical findings discussed later in the paper.

When $\lambda = 0.25$, using (8), the time-varying cessation risk-adjusted discount rate, $\hat{r}$, is found to be 18%; as before, substituting this value of $\hat{r}$, instead of r, in (1) yields the NPV of cash flow to be \$685, which is more than 20% lower than cessation risk-unadjusted NPV in Scenario A. By contrast, when $\lambda = 0.95$, the cessation risk-adjusted discount rate is 15.6%, and the NPV is \$819, which is only 4.6% lower than the CAPM-based NPV in Scenario A.

The results in Table 1, although purely illustrative, underscore the importance of correctly estimating the time profile of cessation risk. On the one hand, ignoring cessation risk when such risk exists can greatly overstate the NPV in a DCF model. On the other hand, the assumption of a constant cessation risk, when in reality this risk is steadily and rapidly declining over time (as is the case when $\lambda = 0.95$), can yield erroneous and artificially low NPV. For example, when $\lambda = 0.95$ in Scenario C, the NPV is \$819; by contrast, if we assume a time-invariant cessation risk of 10%, the resulting NPV of \$416 is nearly 50% lower. In sum, in DCF models, estimating the correct time profile of cessation risk matters considerably.

2.4 Terminal Value in DCF Models

In most practical valuation settings, the input assumptions of the DCF model differ between the first set of explicitly modeled years (typically five to ten years) and the infinite horizon from which the terminal value (TV) is derived. For example, the assumed cash flow growth rate in the explicitly modeled early years is typically higher than that in the infinite horizon. The NPV of the cash flows for the period beyond the explicitly modeled years, which is the TV in DCF models, is discounted back to the valuation date using the appropriate discount rate.

The expression for the TV in a typical DCF model is given by:

$$\frac{(1+g)}{(r-g)},$$

as shown in (1). With cessation risk, the expression for TV is given by (7). However, we can simply use:

$$\text{TV} = \frac{(1+g)}{(\hat{r}-g)}$$

to represent the TV, where $\hat{r}$, given by (8), is the cessation risk-adjusted discount rate, and g denotes the cash flow growth rate in the infinite horizon.

In empirical applications where data strongly suggest that the cessation risk is *not* time invariant (i.e., $\lambda \neq 0$), it is important to recognize that the estimated $\hat{r}$ applies to the discount rate in the infinite horizon, that is, in the computation of TV. For the explicitly modeled years, one should use the cessation rates estimated from the data to compute the appropriate time-varying discount rate for each year. In particular, for any explicitly modeled year i, to compute the risk-adjusted discount rate, we set the PV of cash flow in the traditional DCF model equal to the expected PV of cash flow in the risk-adjusted model:

$$\left[\frac{(1+g)}{(1+\hat{r}_i)}\right]^i = (1-\lambda_0 e^{-\lambda \cdot i}) \cdot \left[\frac{(1+g)}{(1+r)}\right]^i, \tag{10}$$

and then solve for the time-varying discount rate for that year:

$$\hat{r}_i = (1+r)e^{\lambda} \cdot (e^{\lambda \cdot i} - \lambda_0)^{(-1/i)} - 1. \tag{11}$$

In the next section, an empirical application illustrates the usage of $\hat{r}$ to compute TV and the individual $\hat{r}_i s$ for the explicitly modeled years.

3. EMPIRICAL APPLICATION

In this section, we illustrate our model using an empirical application. We use what we consider to be a relevant data set created for this paper using information on start-up firms. Using this data set, we estimate the time-varying survival rate set out in (5) utilizing the estimation equation in (6). We then examine the risk-adjusted discount rates implied by the parameter estimates.

3.1 The Data on Start-Up Firms

In this section, we use data from Dow Jones & Company's VentureSource Database to estimate the mortality risk of start-up firms across various industries, including the Information Technology industry. All the start-up firms in this database were backed either by venture capital or private equity. Thus, these firms meet the market test of having raised funds from professional investors. A variety of information relating to these firms' financing history and schedule is recorded in each firm's profile and frequently updated in the database. The data set used in this paper is current as of 2010. Companies that are in bankruptcy or have gone out of business are not removed from the database but instead, their profiles remain "frozen" with information from the last time data was collected prior to cessation.

We downloaded data on relevant fields to create a data set with information on Date Founded and Ceased Operations Date. Additionally, we captured firm-level information on Industry, Ownership Status, Business Status, State, and Country. We restricted our sample to U.S. firms founded between 1987 and 1999. Firms that were founded after 1999 may be subject to a truncation problem. By restricting the last inception year to be 1999, we allow for the firms to be potentially observed at least for 11 years (since the data set ends in 2010); and for firms founded in 1987, we potentially have 23 years of data. Finally, we excluded from the data set any firms that were categorized as "Acquired/Merged" or "Assets Acquired" because acquisitions could be indicative of either successful or unsuccessful exits. In Table 2 below, we report the number of firms in our data set formed in each of the years, 1987 through 1999.

Table 2: Number of Start-Up Firms by Inception Year

Inception Year	All Firms	IT Firms	Non-IT Firms
1987	101	37	64
1988	113	55	58
1989	123	55	68
1990	140	58	82
1991	152	73	79
1992	189	65	124
1993	201	82	119
1994	289	109	180
1995	376	154	222
1996	454	184	270
1997	455	201	254
1998	623	253	370
1999	1,111	451	660
	4,327	**1,777**	**2,550**

We assumed a firm to have had an "unsuccessful" exit (cessation) if its "Ownership Status" is recorded as "In Bankruptcy" or "Out of Business" in the database. The firms' attrition rates were calculated as follows. For each inception year between 1987 and 1999, we followed the cohort of firms that was formed in that year and observed the number of exits for that cohort in the subsequent years. Firms formed in 1987 have, at most, 23 years of operating history, that is, in terms of the estimation model in (6), T = 1, 2,…, 23; for the firms formed in 1988, T = 1, 2,…, 22, etc.

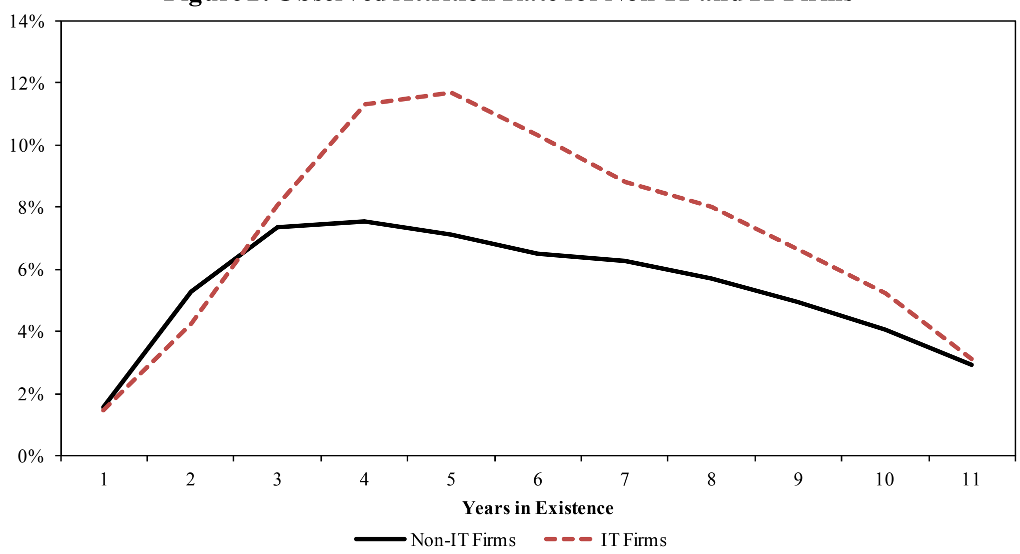
We then computed the time-varying attrition rate for the firms for each inception year cohort. For example, assume for a particular inception year there were N start-up firms. Then for any subsequent year j, the attrition rate is given by:

$$d(j) = e_j \Big/ \left(N - \sum_{k=1}^{j-1} e_k \right), \tag{12}$$

where e_j denotes the number of exits in year j. Note the denominator in the expression in (12) is the number of surviving firms at year j from the initial cohort of N firms. Thus, $d(j)$ provides the conditional probability of cessation, which is then estimated using the transformation set out in (6).

In Figure 2, we depict the observed average attrition rate of non-IT firms and IT firms. The average attrition rate is computed as the average across all firm cohorts of inception years, 1987 through 1999.

Figure 2: Observed Attrition Rate for Non-IT and IT Firms



The attrition rates in Figure 2 suggest an inverted U-shaped pattern: the rate rises for the first four or five years and then steadily declines after that. This pattern accords with intuition. For a venture capital or private equity-backed start-up, if the firm does not show good prospect after the first few formative years, the financial backers are likely to “pull the plug,” and the firm’s funding dries up. Conversely, it appears from the data that after the critical first four or five years, the likelihood of cessation decreases steadily with each successive years of existence.[2] Clearly, the data does not support an assumption of time-invariant attrition.

Figure 2 also shows that, not unexpectedly, the subset of IT firms has higher attrition rates than non-IT firms in the sample. Many of the IT firms in our sample are internet start-ups, mostly with inception years in 1998 and 1999.

3.2 Estimation Results

We have used the data set on start-ups to estimate the model set out in (6). First, we estimated this model for all inception years and all years of existence. For both, non-IT firms and IT firms, the null hypothesis of $\beta = 0$ (which is equivalent to $\lambda = 0$) is strongly rejected by data. The results suggest, as is evident from Figure 2 as well, that cessation probabilities are not time-invariant. As discussed earlier, when the data reject the hypothesis of time-invariance of cessation rates, we cannot use the risk-adjusted discount rate in (8) that is applicable for TV computation. For the explicitly modeled early years, one has to use the estimated risk-adjusted discount rates that *vary by year*.

To illustrate our approach, in what follows, we assume that our data set would be used to estimate the discount rates for a DCF model that has five explicitly modeled years. Thus, $\hat{r}$ would be used to compute TV for the period year six and beyond. We discuss below how we use the estimation results to compute $\hat{r}$ for TV as well as the individual discount rates ($\hat{r}_i s$) for the five explicitly modeled years.

In Table 3 below, we report the results using data on firms that have survived five years or more; that is, in terms of the estimation model in (6), we have used the subset of the data for which $T > 5$.

[2] However, the survival rate may be overstated since firms can continue to exit unsuccessfully even after achieving IPOs.

Table 3: Empirical Results for the Estimation of Cessation Probabilities

	Non-IT Firms		IT Firms	
Parameter	Estimate	T-Statistic	Estimate	T-Statistic
β_0	-1.0617	-3.44	-0.5354	-1.50
β	-0.2690	-11.18	-0.2992	-10.77
λ_0	0.8000		1.7918	
λ	0.2690		0.2992	
Implied Risk-Adjusted Discount Rate ($\hat{r}$)	18.0%		23.4%	

Table 3 reports the parameter estimates (and the associated t-statistics) from the model set out in (6). The t-statistics on the estimates for β strongly reject the null that it is zero and suggests that the cessation rate is time-dependent, a finding consistent with the time profile in Figure 2. The table also shows the results for the estimates for λ_0 and λ implied by the regression results. These $\hat{\lambda}_0$ and $\hat{\lambda}$ are then inputted into (8) to yield the risk-adjusted discount rate ($\hat{r}$) for non-IT firms (= 18.0%) and for the subset of only IT firms (= 23.4%). In computing $\hat{r}$, in both cases we have assumed g = 3% and r = 15%. Since technology firms are likely to have higher "systematic risk" than non-technological sectors of the market, it is certainly conceivable that the CAPM-based discount rate (= r) for IT firms would be higher than that for non-IT firms. If adjustment is also made for market risk, then the risk-adjusted discount rate for IT firms would be higher than the 23.4% shown in Table 3.

As noted earlier, these estimated discount rates should be used in the TV calculation (i.e., $[1+g]/[g-\hat{r}]$) within the DCF framework. However, for the explicitly modeled first five years, it would be incorrect to use these discount rates because these rates are based on parameter estimates from data on firms that have survived five or more years. For the first five years, one has to use the expression for the risk-adjusted time-varying discount rates set out in (11).

With this objective, we first estimated the model parameters (λ_0 and λ) using data on firms during the first five years after inception (i.e., using the data subset for which $T \leq 5$). The parameter estimates for λ_0 and λ and the implied discount rates (which are the $\hat{r}_i s$) using (11) are reported in Table 4. Consistent with the fact that the attrition rates increase during the first five years (see Figure 2), we observe that the discount rates rise as well. They range between 16.5% and 20.2% for non-IT firms and 16.3% and 24.7% for IT firms. Thus, when applied to a DCF model, one would use these $\hat{r}_i s$ to calculate the PV for the first five years and use the $\hat{r}$ in Table 3 for the TV calculation.

Table 4: Estimated Parameters and the Implied Discount Rates

Parameter	Non-IT Firms	IT Firms
λ_0	0.0062	0.0048
λ	-0.6929	-0.8483
Year 1	16.5%	16.3%
Year 2	16.5%	16.5%
Year 3	17.0%	17.4%
Year 4	18.1%	19.5%
Year 5	20.2%	24.7%

We now illustrate the valuation implications of these discount rates using a stylized and simple DCF framework. The details of this DCF model are contained in Appendix 1. Assume in this model that the cash flow in Year 1 is $100, and it grows at 10% per annum for the first five years and then at 3% for years six and beyond. Also assume the cessation risk-*unadjusted* discount rate is 15%, which in a traditional DCF model is the same for all years. These inputs yield a NPV of $1,083 in the DCF model.

Now assume instead one uses the risk-adjusted discount rates for the IT firms shown in Table 4 for the first five years and a 23.4% (i.e., $\hat{r}$ from Table 3) discount rate for TV. Keeping the growth rate assumptions unchanged, usage of these risk-adjusted discount rates yields a NPV of $678, which is nearly 37% lower than the NPV using a CAPM-based discount rate of 15%.

3.3 Illustration Using Hedge Fund Data

We estimate cessation risk with a hedge fund data set constructed from the Hedge Fund Research Database and the TASS Database. Using the same procedures that were applied to the start-up data set, we first calculated hedge fund attrition rates over time. For the purpose of this paper, a fund that stops reporting returns *and* experiences a negative cumulative return for the last three months prior to stoppage of reporting is considered to have ceased operations. We then estimated the model in (6) using these data for hedge funds formed between 1995 and 2008.

The results are reported in Table 5, and they are markedly different from those in Table 3, using the data on start-ups. In particular, the null hypothesis $\beta = 0$ is clearly not rejected since the t-statistic is 0.09. This result implies absence of time-dependence in attrition rates; that is, the risk of cessation for a hedge fund remains high regardless of its age. This is not surprising; the highly-leveraged "bets" made by hedge fund managers may fail at any time period. Thus, a time-invariant attrition rate is appropriate for the hedge fund industry, and based on $\hat{\lambda}_0$ reported in Table 5, this rate is nearly 7%. If, as before, one were to assume that the unadjusted discount rate, *r,* is 15%, then substituting $\hat{\lambda}_0 = 7\%$ into the time-invariant formulation:

$$r^* = \frac{(\hat{\lambda}_0 + r)}{(1 - \hat{\lambda}_0)},$$

one obtains a cessation risk-adjusted discount rate of 23.6%. Using the same cash flow assumptions used in the illustrative DCF framework in the preceding subsection, this discount rate yields a NPV of $655, which is lower than the NPV of IT start-up firms.

Table 5: Estimation Results from Hedge Fund Data

Parameter	Estimate	T-Statistic
β_0	-2.8618	-20.38
β	0.0018	0.09
λ_0	0.0695	
Implied Discount Rate	23.6%	

3.4 A Generalization to Non-Monotonic Survival Function

In the interest of tractability and simplicity, we have adopted a form for cessation risk that is a monotonic function of time. In Appendix 2 of the paper, we propose a generalization of the form which accommodates a wide variety of time profiles for cessation risk, including the monotonic form used in the paper. This generalized form comes at a cost; the expression for the risk-adjusted discount rate involves the polylogarithm function, which is not readily evaluated using standard spreadsheet software such as Excel. However, estimation results using the data on start-ups suggest that beyond the fifth year, the cessation risk *does* decline monotonically. That is, the data set used in this paper rejects the need for a more general form for time-varying cessation risk. In applications to other data sets however, if one finds a non-monotonic time profile for cessation risk, then the generalization proposed in Appendix 2 would prove useful.

4. Concluding Comments

The principal contribution of this paper is to propose a tractable functional form for the cash flow cessation probability such that it yields a closed-form solution for NPV in a DCF model. Application to a new data set created for this paper shows that venture capital-backed start-up firms in the U.S. exhibit time-varying cessation risk: the risk increases during their first five years of existence and then declines thereafter.

These findings suggest that the assumption of no cessation risk, as the CAPM posits, or a time-invariant cessation risk, would both be erroneous. Indeed, because the cessation risk changes over time, so should the discount rate used to value these firms. Usage of time-varying discount rates estimated from the data yield NPVs that are significantly lower than the NPV based on cessation risk-unadjusted discount rate within a DCF model.

APPENDIX 1: IMPLICATIONS FOR VALUATION WITHIN A DCF FRAMEWORK

Panel A:

Assumptions

Discount Rate	15.0%
Cash Flow Growth Rate	10.0%
Terminal Growth Rate	3.0%

	Year 1	Year 2	Year 3	Year 4	Year 5	
Projected Flow	$100	$110	$121	$133	$146	
Terminal Value						$1,257
Discount Rate	15.0%	15.0%	15.0%	15.0%	15.0%	
Present Value Factor	1.00	0.87	0.76	0.66	0.57	0.50
Present Value of Cash Flows	$100	$96	$91	$88	$84	$625

Total Present Value **$1,083**

Panel B:

Assumptions

Terminal Risk-Adjusted Discount Rate	23.4%
Cash Flow Growth Rate	10.0%
Terminal Growth Rate	3.0%

	Year 1	Year 2	Year 3	Year 4	Year 5	
Projected Flow	$100	$110	$121	$133	$146	
Terminal Value						$738
Cessation Risk-Adjusted Discount Rate	16.3%	16.5%	17.4%	19.5%	24.7%	
Present Value Factor	1.00	0.86	0.73	0.59	0.41	0.35
Present Value of Cash Flows	$100	$94	$88	$78	$61	$258

Total Present Value **$678**

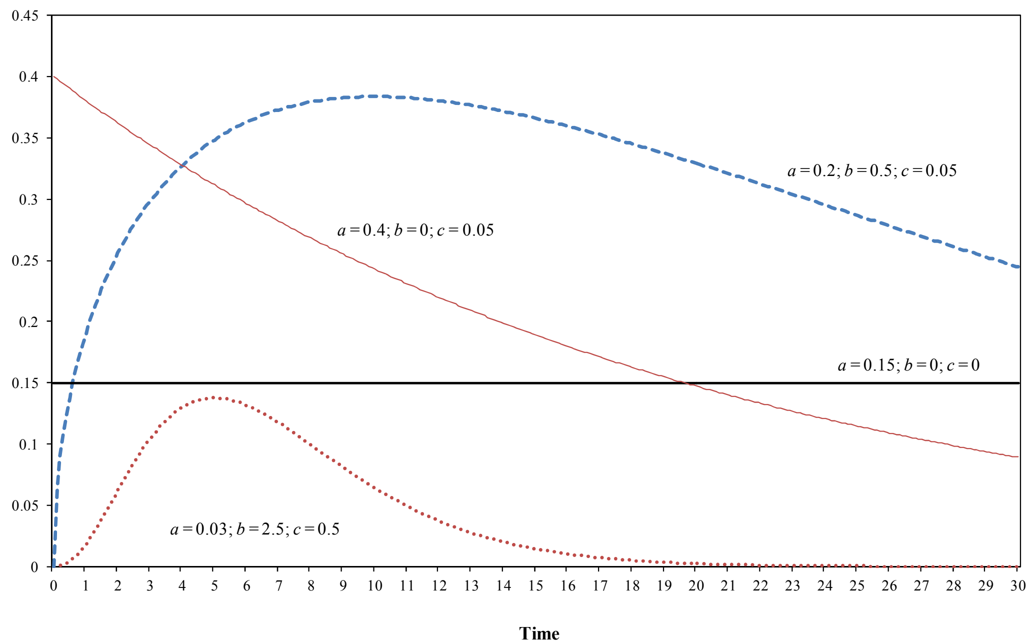
APPENDIX 2: A GENERALIZATION TO NON-MONOTONIC SURVIVAL FUNCTION

We can generalize the survival function, (5), discussed in the main body of the paper, as:

$$s(i) = 1 - a \cdot i^{b} \cdot e^{-c \cdot i}. \tag{A1}$$

Thus, the time-varying attrition rate is: $d(i) = 1 - a \cdot i^{b} \cdot e^{-c \cdot i}$, where a, b, and c are non-negative parameters. In Figure 3 below, we illustrate the attrition rates for various parameter values; the figure shows that the functional form can accommodate a wide variety of shapes for the time profile of attrition rates.

Figure 3: Attrition Rates Under Alternative Parameter Values



As before, (A1) subsumes no cessation risk ($a = 0$) and time-invariant cessation risk ($a > 0$; $b = c = 0$) as special cases. However, unlike (5), the more general form (A1) can be non-monotonic with respect to time. Differentiating $d(i)$ with respect to i and setting the resulting expression equal to zero, we find the $d(i)$ reaches a maximum when $i = b / c$. Thus, (5), the monotonic version discussed in the main body of the paper, is a special case of (A1) when $b = 0$.

Denoting $i^* = b/c$, it is readily verified that:

$$d(i^*) = a\left(\frac{b}{c}\right)^b e^{-b}, \tag{A2}$$

which denotes the maximum value of $d(i)$.

The NPV using (A1) is found to be:

$$V_4 = \sum_{i=1}^{\infty}(1 - a \cdot i^b \cdot e^{-c \cdot i}) \cdot \left(\frac{1+g}{1+r}\right)^i = \frac{1+g}{1+r} - a\mathrm{Li}_{-b}\left(e^{-c}\frac{(1+g)}{(1+r)}\right), \tag{A3}$$

where:

$$\mathrm{Li}_m(n) = \sum_{j=1}^{\infty}\frac{n^j}{j^m}$$

is the polylogarithm function. This function, while readily quantifiable in softwares such as Mathematica or Matlab, requires some programming within Excel because it is not an Excel built-in function.

Let us denote:

$$z = \mathrm{Li}_{-b}\left(e^{-c}\frac{(1+g)}{(1+r)}\right);$$

then the cessation risk-adjusted discount rate is given by:

$$\hat{\hat{r}} = \frac{(1+g)\cdot r + a \cdot g \cdot (g-r)\cdot z}{(1+g) + a\cdot(g-r)\cdot z}. \tag{A4}$$

In the absence of cessation risk, i.e., when $a = 0$, it follows from (A4) that $\hat{\hat{r}} = r$.

Finally, the parameters a, b, and c in (A1) can be estimated by noting that:

$$ln[1 - s(i)] = ln[d(i)] = ln(a) + b\,ln(i) - c \cdot i,$$

which yields the following estimation equation:

$$y_t = \alpha_0 + \alpha_1 ln(T_t) + \alpha_2 T_t + \varepsilon_t. \tag{A5}$$

Thus, $\hat{a} = e^{\hat{\alpha}_0 + \hat{\sigma}^2/2}$; $\hat{b} = \hat{\alpha}_1$ and $\hat{c} = -\hat{\alpha}_2$.

We have estimated the model in (A5) using the data set on start-up firms. The estimation results do not reject the null hypothesis $\alpha_1 = 0$; this implies $b = 0$ in (A1). Thus, data support the usage of the more restrictive survival functional form in (5) in the main body of the paper.

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